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LETTER TO THE EDITOR

Oscillating Maxwellians

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Abstract. We obtain explicit homogeneous and inhomogeneous solutions for the d-dimensional nonlinear Boltzmann equations. We assume Maxwell particles and time dependent external forces proportional either to the velocity or to the space variables. We find examples of distributions which relax towards oscillating Maxwellians and others which relax towards absolute Maxwellians.

At the present time, it is of interest to study asymptotic behaviour of physical models subjected to external forces with a varying parameter. The study of the equilibrium distributions of the Boltzmann equation, with external forces spatially, $\boldsymbol{x}$, dependent but without velocity, $\boldsymbol{v}$, is very old, Boltzmann himself contributed to this study (Boltzmann 1909, Uhlenbeck and Ford 1963, Cergignani 1975). Assuming a vanishing collision term, he found time dependent solutions.

Here we study first $a_1(t)\boldsymbol{v}$ forces and are mainly interested in oscillating $a_1(t)$ with a varying parameter. We give explicit homogeneous and inhomogeneous solutions with a non-trivial collision term. The homogeneous equilibrium distributions can oscillate between different Maxwellians multiplied by an oscillating time factor, while the inhomogeneous ones to go to zero (expansion) and we discuss the possibility of introducing sources. Second, with $a_1(t)\boldsymbol{x}$ forces or harmonic external potentials, we still find the possibility of oscillating Maxwellians.

We start with a d-dimensional Boltzmann equation (BE)

$$(\partial_t + \boldsymbol{v}\cdot\partial_x + \boldsymbol{A}_0(t)\cdot\partial_v + \Lambda(\boldsymbol{v}, \boldsymbol{x}, t))f(\boldsymbol{v}, \boldsymbol{x}, t) = \mathrm{Col}(f)$$

$$\Lambda = a_1(t)\partial_v\cdot\boldsymbol{v} \qquad \text{or} \qquad \Lambda = a_1(t)\boldsymbol{x}\cdot\partial_v \tag{1}$$

and assume Maxwellian particles. In the nonlinear collision term $\mathrm{Col}(f)$, the interaction appears only through the scattering cross section $\sigma^{(d)}(\chi)$. $d=1$ is the Kac model (Kac 1956, Uhlenbeck and Ford 1963), for which the momentum has been dropped; $d\geqslant 2$ models for which energy and momentum conservation holds; $d=3$ is the standard Boltzmann model. When no external force is present, explicit solutions are known (Ernst 1979, 1981, Cornille and Gervois 1980) which are the generalisation of the $d=3$ so called Bobylev-Krook-Wu solution (Bobylev 1976, 1984, Krook and Wu 1976, Muncaster 1979). For the Kac model, other exact solutions exist (Cornille 1984, 1985a, b, c).

Here we give new exact solutions for $d\geqslant 1$ and study their relaxation towards equilibrium. We sketch briefly the results, while a complete study will be performed elsewhere (Cornille 1985d).

We begin with $a_1(t)\boldsymbol{v}$ forces and $d \geq 2$ homogeneous formalism. Let us assume that for $t > t_0$, $\mathrm{Col}(f) \simeq 0$. Then the LHS of (1) defines an asymptotic solution f_{as}:

$$f_{as}(\boldsymbol{c}, t) \simeq \text{constant } \nu \exp[-(\gamma \boldsymbol{c})^2/2] \qquad \gamma = \nu^{1/d} = \exp\left(-\int_{t_0}^{t} a_1(t')\, \mathrm{d}t'\right) \qquad t > t_0$$

$$\langle \boldsymbol{v} \rangle \gamma(t) = \langle \boldsymbol{v} \rangle_{t=0^+} \int_{t_0}^{t} \boldsymbol{A}_0(t') \gamma^{-1}(t')\, \mathrm{d}t' \qquad \boldsymbol{c} = \boldsymbol{v} - \langle \boldsymbol{v} \rangle \tag{2}$$

$\langle \boldsymbol{v} \rangle$ being the mean velocity, $\rho_0(t) = \rho_0(0)$, the local density $\rho_0 = \int f\, \mathrm{d}\boldsymbol{v}$ and $\boldsymbol{c}$ the peculiar velocity. f_{as} is the product of a time factor ν by a $(\boldsymbol{c}, t)$ dependent Gaussian. We find (i) $\int^t a_1\, \mathrm{d}t' \to \pm\infty, f_{as} \to 0$; (ii) $\int^t a_1\, \mathrm{d}t' \to \text{constant}, f_{as} \to \exp(-c^2) \times \text{constant}$; (iii) $\int^t a_1\, \mathrm{d}t'$ oscillates, f_{as} oscillates too.

In the last case, we discuss the Gaussian $(\boldsymbol{c}, t)$ term of (2). For instance, we assume that a_1 is periodic and more precisely

$$\int_0^t a_1\, \mathrm{d}t' = \sin t + \lambda \sin qt \tag{3}$$

(for simplicity we choose $t_0 = 0$). We obtain (i) if $\lambda = 0$, $\gamma(t)$ is periodic with period $T = 2\pi$ and the Gaussian term will oscillate between two Maxwellians; (ii) if $\lambda \neq 0$ and q integer, we have competition between a circular function $T_1 = 2\pi$ and a harmonic function $T_2 = 2\pi/q$. For instance, for $q = 2$, $0 < \lambda < 0.5$, the Gaussian still oscillates between two Maxwellians and four for $\lambda > 0.5$. For $q = 3$, $0 < \lambda < \frac{1}{9}$ we find two Maxwellians, four for $\lambda > \frac{1}{9}$ and three for $\lambda = 1$. And so on for $q = 4, 5, \ldots$. (For q irrational, the Gaussian term is quasi-periodic with a countable set of extremal Maxwellians.) Adding the trivial periodic time factor ν in f_{as}, $\partial_t \log f_{as} \simeq a_1(-d + |\boldsymbol{c}\gamma|^2)$, the extrema are no longer provided only by $a_1(t) = 0$ but also by $|\boldsymbol{c}|$, t values.

The important point is that starting with a force, linear superposition of two circular functions, we emerge with f_{as} being the product of two periodic functions. When the parameter is varying we observe the appearance of possible harmonics. Here the asymptotic behaviour, being obtained with negligible collision terms (the nonlinear part of the BE), are provided by the linear part of the BE.

In the inhomogeneous formalism, $\mathrm{Col}(f) \simeq 0$ for $t > t_0$, we obtain a particular class $f_{as} \simeq \nu \exp[-\frac{1}{2}(\gamma \boldsymbol{c})^2]$, ν given by (2) but γ is different:

$$\gamma(t) = \left[1 + \mu_0 \int_{t_0}^{t} \exp\left(\int_{t_0}^{t'} a_1\, \mathrm{d}t''\right) \mathrm{d}t'\right] \exp\left(-\int_{t_0}^{t} a_1\, \mathrm{d}t'\right) \tag{2'}$$

and reduces to the previous one if the constant μ_0 is zero. For $\langle \boldsymbol{v} \rangle = \boldsymbol{v} - \boldsymbol{c}$, a new term $\mu_0 \boldsymbol{x}$ appears, while $\rho_0 = \nu\gamma^{-d}$. This class is particular because ρ_0 is $\boldsymbol{x}$ independent and $\boldsymbol{x}$ is present only in $\langle \boldsymbol{v} \rangle$ and linear. If, as in the homogenous case, we study the asymptotic behaviour in terms of the forces $a_1(t)$, then a great difference occurs. In order to have non-trivial ($\neq 0$ or $\neq \infty$) f_{as} when $t \to \infty$, both terms $\nu(t)$ and the Gaussian must be non-trivial. Let us assume

$$\alpha_{\inf} < \nu^{-1/d} = \exp \int_{t_0}^{t} a_1\, \mathrm{d}t' < \alpha_{\sup}$$

$\alpha_{\sup}$ and $\alpha_{\inf}$ being finite constants. ν is then non-trivial but $(1 + \mu_0 \alpha_{\inf} t)\alpha_{\sup}^{-1} < \gamma(t) < (1 + \mu_0 \alpha_{\sup} t)\alpha_{\inf}^{-1}$. Consequently $|\gamma \boldsymbol{c}| \to \infty$ and $f \to 0$. In conclusion, we can have separately a non-trivial time factor ν or a Gaussian term, but not both.

In the rest of this letter we present exact solutions relaxing towards f_{as} given by (2) or (2′) and at the end, solutions for $a_1\boldsymbol{x}$ forces.

In table 1(a) we present the results for the homogeneous $d=1$ Kac model with linear operator $\partial_t + a_0(t)\partial_v + a_1(t)\partial_v v$.

Table 1. Homogeneous Kac model $d=1$, $a_0(t)\partial_v + \Lambda(v, t)$.

$$f(\omega, t) = \frac{\exp[-\omega^2/2(1-\eta)]}{[2\pi K(1-\eta)]^{d/2}}\rho_0(0)\nu(t)\left[1+\frac{1}{2}\frac{\bar{\eta}}{(1-\eta)}\left(\frac{\omega^2}{1-\eta}-1\right)+\omega\frac{\bar{\bar{\eta}}}{1-\eta}\right] \qquad \omega = \frac{v\exp(-\int_0^t a_1(t')\,dt')}{\sqrt{K}}$$

$$f_{as} = \frac{\exp(-\omega^2/2)}{(2\pi K)^{1/2}}\rho_0(0)\nu(t) \qquad \frac{a_0(t)}{a_0(0)} = \left(\frac{\rho_0(t)}{\rho_0(0)}\right)^2 \exp\left(-\tau_F\int_0^t \rho_0(t')\,dt'\right)$$

$$\bar{\bar{\eta}}(t) = \frac{a_0(0)\exp(-\tau_F\int_0^t \rho_0(t')\,dt')}{K^{1/2}\rho_0(0)(\tau_3-\tau_1+\sigma_2)}$$

$$\varphi(t) = \varphi(0)\exp\left(-\sigma_2\int_0^t \rho_0(t')\,dt'\right) \qquad \eta(t) = \varphi(t) + \bar{\bar{\eta}}^2\frac{(\tau_3-\tau_1+\sigma_2)}{\tau_F(2\tau_F-\sigma_2)}\sigma_2 \qquad \bar{\eta}(t) = \eta + \frac{\bar{\bar{\eta}}^2(\tau_1-\tau_3-\sigma_2)}{\tau_F}$$

$$\tau_m = \int_{-\pi}^{+\pi}(\cos x)^m\sigma^{(1)}(x)\,d\chi \qquad \sigma_2 = \tau_2-\tau_4 \qquad \tau_F = \tau_0-\sigma_2-\tau_3 > 0 \qquad 2\tau_F-\sigma_2 > 0$$

$\rho_0(0)$, K, $\varphi(0)$, $a_0(0)$ constants such that $f(\omega, 0) > 0$; if $a_0(0) = 0 \rightarrow \bar{\bar{\eta}} = 0 \rightarrow \bar{\eta} = \eta = \varphi$

(a) $\Lambda = a_1\partial_v v$ $\qquad \nu(t) = \exp\left(-\int_0^t a_1\,dt'\right)$ $\qquad \rho_0(t) = \rho_0(0)$

(b) $\Lambda = a_1 v\partial_v$ $\qquad \nu(t) = 1$ $\qquad \rho_0(t) = \rho_0(0)\exp\left(\int_0^t a_1\,dt'\right)$

If $a_0 \equiv 0$, the exact solution is necessarily even in v, while if $a_0 \neq 0$ an odd part exists, but $a_0(t)$ and $a_1(t)$ are linked. As in (2) or (2′) the solution is the product of a $\nu(t)$ factor by a function of t, v. f can be written:

$$f\left[\omega = vK^{-1/2}\exp\left(-\int^t a_1\,dt'\right), t\right],$$

K being a constant. η, $\bar{\eta}$, $\bar{\bar{\eta}}$ depend on both the moments τ_m of the differential cross section $\sigma^{(1)}(x)$ and t. η, $\bar{\eta}$, $\bar{\bar{\eta}} \rightarrow 0$ when $t\rightarrow\infty$, while $\rho_0(t) = \rho_0(0)$.

We obtain (i) if $\int^t a_1\,dt' \rightarrow \pm\infty$, $f\rightarrow 0$; (ii) if $\int^t a_1\,dt' \rightarrow$ constant, $f\rightarrow a$ Maxwellian; (iii) if $\int^t a_1\,dt'$ oscillates between two finite values,

$$f\rightarrow f_{as} = (2\pi K)^{-1/2}\rho_0(0)\left[\exp\left(-\int_0^t a_1\,dt'\right)\right]\exp(-\omega^2/2)$$

i.e. a product of a pure oscillating time term by an oscillating Maxwellian.

In the last two cases there exist two successive regimes: first, up to the t_0 value of (2) where η, $\bar{\eta}$, $\bar{\bar{\eta}} \rightarrow 0$ and f becomes equivalent (but not identical) to f_{as}, and second the oscillatory f_{as} regime. In order to have an estimation of the first regime, we define a reduced distribution $F(\omega, t) = f(\omega, t)/f_{as}(\omega, t)$ and investigate the large t, fixed ω behaviour:

$$F - 1 \simeq -\omega^4/8\eta^2(t) + \omega\bar{\bar{\eta}}(t). \tag{4}$$

We associate two relaxation times $T_e = (2\sigma_2\rho_0(0))^{-1}$, $T_0 = (\tau_F\rho_0(0))^{-1}$ for the even and odd parts. Depending on whether $T_0 < T_e$ or $T_e < T_0$, the relaxation is from below or can be from above, so that the relaxation towards f_{as} depends on the model of the cross section. We have numerically checked these properties (Cornille 1985d).

In table 2(a), for $d \geq 2$ we have written down a class of homogeneous ($\mu_0 = 0$) and inhomogeneous ($\mu_0 \neq 0$) solutions $f(\boldsymbol{\omega}, t)$ with $\boldsymbol{\omega} = \gamma(t)\boldsymbol{c}K^{-1/2}$, K being a constant and γ the same as in (2′). We still have a time factor $\nu(t)$ multiplied by a function $g(\boldsymbol{\omega}, t)$ playing a role similar to the above Gaussian:

$$g(\boldsymbol{\omega}, t) = f(2\pi K)^{d/2}(\rho_0(0)\nu)^{-1} \qquad g_{as} = \exp(-\boldsymbol{\omega}^2/2) \tag{5}$$

All the previous properties of the Kac model can be studied. If $\varphi(t) \to 0$, we define a reduced distribution $F = f(\boldsymbol{\omega}, t)/f_{as}(\boldsymbol{\omega}, t) = g(\boldsymbol{\omega}, t)/g_{as}(\boldsymbol{\omega})$, and find $F - 1 \simeq -\omega^4\varphi^2(t)/8$ with a relaxation from below. The relaxation time is $(\sigma_2^{(d)}\rho_0(0))^{-1}$ for $\mu_0 = 0$ and can be estimated for $\mu_0 \neq 0$. If we study the asymptotic behaviour in terms of the $(\boldsymbol{c}, t)$ variable, a great difference occurs between $\mu_0 = 0$ and $\mu_0 \neq 0$. For the homogeneous $\mu_0 = 0$ solutions with local density $\rho_0(t) = \rho_0(0)$ we find (i) $\int_0^t a_1\, dt' \to \pm\infty$, $f \to 0$ ($g \to$ constant or zero); (ii) $\int_0^t a_1\, dt \to$ constant, g and $f \to a$ Maxwellian; (iii) $\int_0^t a_1\, dt'$ oscillating, then g, f oscillate too. Numerically, for $d = 3$ homogeneous $g(c, t)$ we choose the forces $a_1(t)$ of (2) with $q = 3$. In figure 1 for the two values $\lambda = 0.1, 1$ we plot the different relaxation curves. We define a new time variable $\tau\pi = t$. We see that after $\tau = 0.4, 0.4$ the asymptotic regimes are reached and $g \simeq g_{as}$ oscillates between two or three Maxwellians.

On the other hand, for the $\mu_0 \neq 0$ inhomogeneous formalism where ρ_0 is time dependent, we find the same difficulty as above in (2′). For instance if $a_1(t) = a_1(0) > 0$, then $\gamma(t) \to \infty$, $g \to 0, f \to 0$, except if $|\boldsymbol{c}| = 0$. If we assume that $\int_0^t a_1\, dt'$ is bounded from below and from above (for instance, $\int_0^t a_1\, dt' \to$ constant or oscillating) then ν is also bounded but $\gamma \to \infty$, g and $f \to 0$ except for $|\boldsymbol{c}| = 0$.

In conclusion, in the presence of $a_1(t)\partial_{\boldsymbol{v}} \cdot \boldsymbol{v}$ oscillating external forces, the particles described by the various d-dimensional exact solutions exhibit general features which are different in the homogeneous and inhomogeneous formalisms. Let us try to give a physical meaning to the force. If we consider that the particles described by the BE are a suspension in viscous fluids (or aerosols in gas) then the force can be described as the decelerations due to viscosity. If we try to understand the failure of these inhomogeneous distributions ($f \to 0$ corresponding to an expansion), it may be true that the particles are not confined in the space because we have not introduced boundary conditions. We could also try to introduce both sources and forces. We present a mathematical model: in (1) let Λ be the sum of a force and a source

$$\Lambda = a_1\boldsymbol{v} \cdot \partial_{\boldsymbol{v}} = a_1\partial_{\boldsymbol{v}} \cdot \boldsymbol{v} - da_1. \tag{6}$$

Then the density is not conserved but in f_{as} provided by (2) or (2′) ν is a constant. The corresponding exact solutions for the Kac model and dimensions $d \geq 2$ are written down in tables 1(b) and 2(b). Now in both homogeneous and inhomogeneous formalisms, oscillating Maxwellians are allowed (Cornille 1985d). In the inhomogeneous $\mu_0 \neq 0$ case we can choose either $a_1(t) = a_1(0) + \partial_t \log(\lambda_0 + \lambda_1 \sin t + \lambda_2 \sin qt)$ or $\gamma(t)^{-1} = 1 + (\sin t + \lambda \sin qt)r$. On the other hand, in table 2(b) ($\mu_0 \neq 0$), let us assume $a_1(t) \equiv a_1(0) > 0$; when $t \to \infty$, $\gamma \to \mu_0$ and $f \to (2\pi K)^{d/2} \exp(-c^2\mu_0^2K^{-1})$, an absolute Maxwellian.

In table 2(c) we have written down explicit $d \geq 2$ inhomogeneous solutions corresponding to an harmonic velocity dependent potential with force $\Lambda = a_1(t)\boldsymbol{x} \cdot \partial_{\boldsymbol{v}}$. Then $\nu(t) = 1$ and the Gaussian still has the variable $\boldsymbol{\omega} = \boldsymbol{c}\gamma(t)K^{-1/2}$ but $a_1(t) = (\partial_{t^2}^2\gamma)\gamma^{-1}$.

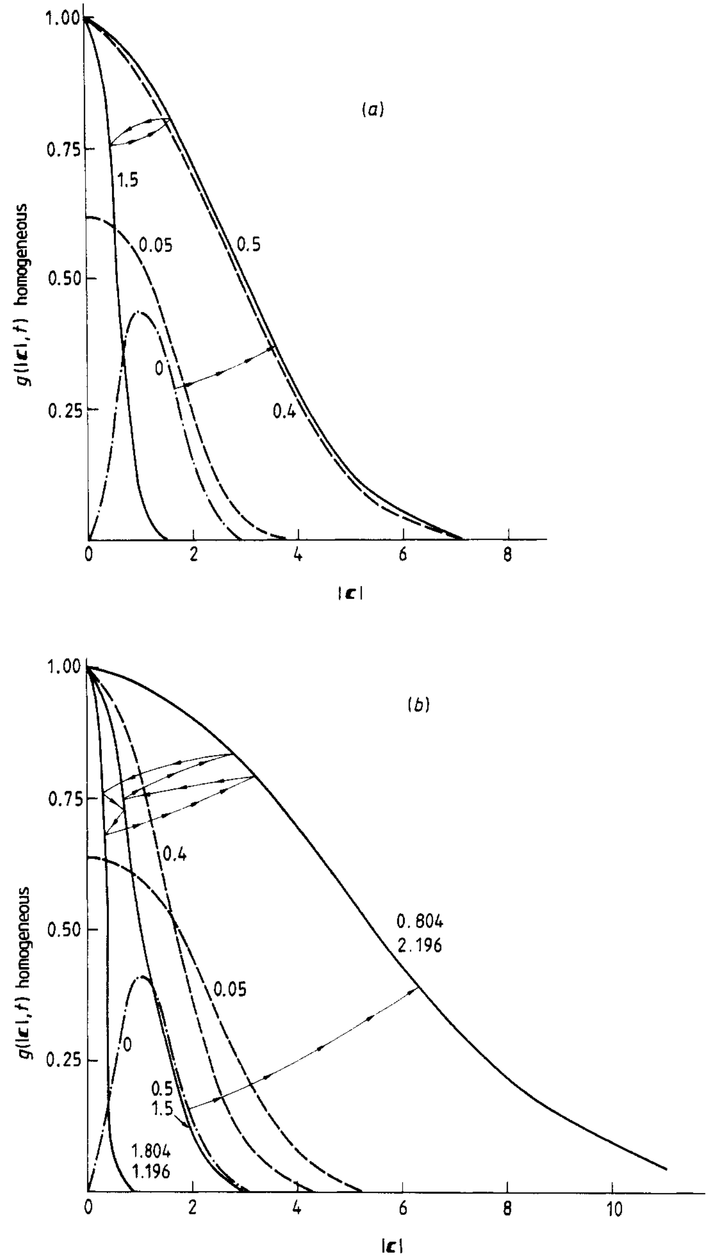

Figure 1. Plot of the homogeneous $g(|c|, t=\tau\pi)$ given by (5) against $|c|$, $a_1(t)=\sin t+\lambda\sin 3t$, $\rho_0(0)=2$, $\varphi(0)=0.4$, $K=1$, $\sigma_2^{(3)}=1$, $d=3$. (a) $\lambda=0.1$, (b) $\lambda=1.0$. Values of τ given on each curve.

We easily find oscillating Maxwellians, for instance with $\gamma=1+(\sin t+\lambda\sin qt)\,r$. We can also have solutions relaxing towards absolute Maxwellians: for instance, with $a_1=a_1(0)\exp(-t)$, $a_1(0)>0$, then $[I_0(2a_1^{1/2}(0))]^{-1}I_0[2a_1^{1/2}(0)\exp(-t/2)]=\gamma(t)$, $\gamma(t)\to$ constant and $f\to\exp(-\text{constant}\times|c|^2)$. We notice that for spatially dependent forces

Table 2. $d \geqslant 2$, $\boldsymbol{A}_0(t) \cdot \partial_{\boldsymbol{v}} + \Lambda(\boldsymbol{v}, \boldsymbol{x}, t)$.

Homogeneous, $\mu_0 = 0$ Inhomogeneous, $\mu_0 \neq 0$

$$f(\boldsymbol{\omega}, t) = \frac{\exp[-\boldsymbol{\omega}^2/2(1-\varphi)]}{[2\pi K(1-\varphi)]^{d/2}}\rho_0(0)\nu(t)\left[1 + \frac{1}{2}\frac{\varphi}{(1-\varphi)}\left(\frac{\boldsymbol{\omega}^2}{1-\varphi} - d\right)\right] \qquad \boldsymbol{\omega} = \frac{\boldsymbol{C}\gamma(t)}{\sqrt{K}} \qquad \boldsymbol{C} = \boldsymbol{V} - \langle \boldsymbol{V} \rangle$$

$$f_{\text{as}} = \frac{\exp(-\boldsymbol{\omega}^2/2)}{(2\pi K)^{d/2}}\rho_0(0)\nu(t) \qquad \langle \boldsymbol{v} \rangle \gamma = \mu_0 \boldsymbol{X} + \int_0^t \boldsymbol{A}_0(t')\gamma(t')\,\mathrm{d}t' + \langle \boldsymbol{v} \rangle_{t=0, \boldsymbol{X}=0}$$

$$\varphi(t) = \varphi(0)\exp\left(-\sigma_2^{(d)}\int_0^t \rho_0(t')\,\mathrm{d}t'\right) \qquad \sigma_2^{(d)} = \int \sin^2 x\sigma_2^{(d)}(x)\,\mathrm{d}\Omega_d\left(\int \mathrm{d}\Omega_d\right)^{-1}$$

$$f > 0 \qquad \text{if } 0 < \varphi(0) < (1 + \tfrac{1}{2}d)^{-1}$$

$$\rho_2(t)\gamma^2 = \rho_0(t)[Kd + \gamma^2\langle \boldsymbol{v} \rangle^2] = \int v^2 f\,\mathrm{d}\boldsymbol{v} \qquad K \text{ (or } \langle \boldsymbol{v} \rangle_{t=0, \boldsymbol{X}=0}),\ \rho_0(0),\ \varphi(0) \text{ constants}$$

$\mu_0 = \text{constant}$	$\gamma(t) = \dfrac{1 + \mu_0 \int_0^t \exp(\int_0^{t'} a_1\,\mathrm{d}t'')\,\mathrm{d}t'}{\exp(\int_0^t a_1\,\mathrm{d}t')}$	$a_1 = \mu_0\gamma^{-1} + \partial_t \log \gamma^{-1}$	
(a) $\Lambda = a_1(t)\partial_{\boldsymbol{v}} \cdot \boldsymbol{v}$	$\nu(t) = \exp\left(-d\int_0^t a_1(t')\,\mathrm{d}t'\right)$	$\rho_0(t) = \rho_0(0)\nu\gamma^{-d} = \rho_0(0)$	if $\mu_0 = 0$
(b) $\Lambda = a_1(t)\boldsymbol{v} \cdot \partial_{\boldsymbol{v}}$	$\nu(t) = 1$	$\rho_0(t) = \rho_0(0)\gamma^{-d}$	
$\mu_0 = \partial_t \gamma$	$\gamma(0) = 1$	$\partial_t^2 \gamma = a_1(t)\gamma$	
(c) $\Lambda = a_1(t)\boldsymbol{x} \cdot \partial_{\boldsymbol{v}}$	$\nu(t) = 1$	$\rho_0(t) = \rho_0(0)\gamma^{-d}$	

other exact solutions exist. To my knowledge, the examples given in this letter of $d \geqslant 2$ inhomogeneous solutions relaxing towards absolute Maxwellians, are the first explicitly known nonlinear BE solutions having that property.

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